Some Fixed Point Results for Generalized $(\alpha\mathfrak{f} - \mathrm{Q}\varphi)$ −Contractive Mappings via Binary Relation in Metric Spaces

Aarju and Anil Kumar
Department of Mathematics,
School of Physical Sciences,
Starex University, Gurugram,
122413, Haryana, India.

Abstract In this manuscript, we expand the concept of generalized altering distance function and introduced a generalized $(\alpha\mathfrak{f} - \beta\varphi)$ −contractive mappings and give fixed point theorems for such contractions in metric spaces. Some existing results of literature are also provided which are direct consequences of our main results.

Keywords complete metric space, fixed point, generalized altering distance function.
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1. Introduction and Preliminaries

Fixed point theory is a branch of mathematics that deals with the study of mappings that have points that remain unchanged under the mapping process. It provides powerful tools and techniques for analyzing the existence, uniqueness, and stability of fixed points in different settings. Fixed point theory has found applications in diverse fields.

One of the most fundamental theorems in fixed point theory is the Banach fixed point theorem [3], also known as the contraction mapping principle. This theorem has numerous applications in areas such as functional analysis, numerical analysis, differential equations, game theory etc. Stefan Banach valuable work has been built by generalizing the metric conditions or by imposing conditions on the metric spaces (see [1-18]).

In this manuscript, we expand the concept of generalized altering distance function and introduced a generalized $(\alpha\mathfrak{f} - \beta\varphi)$ −contractive mappings and give fixed point theorems for such contractions in metric spaces.

Khan *et al.* [11] use a control function (altering distance function) they referred to as a changing distance function allowed them to tackle new fixed point problems.

"**Definition 1.1. [11]** A function $\mathfrak{f} : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied

(i) $\mathfrak{f}(0) = 0$ if and only if $t = 0$,

(ii) $\mathfrak{f}$ is continuous and monotonically non-decreasing."

We first recall the auxiliary functions that we shall use effectively.

"**Definition 1.2.** Let K be a set, and let R be a binary relation on K. We say that $P : K \to K$ is an R-preserving mapping if

ḩ, ķ $\in K$: ḩRķ $\Longrightarrow$ PḩRPķ."

"**Definition 1.3.** Let $\aleph \in \mathbb{N}$. We say that R is $\aleph$-transitive on K if

$ḩ_0, ḩ_1, \dots, ḩ_{\aleph+1} \in K : ḩ_i R ḩ_{i+1}$, for all $i \in \{0, 1, \dots, \aleph\} \Longrightarrow ḩ_0 R ḩ_{\aleph+1}$."

"**Remark 1.4.** Let $\aleph \in \mathbb{N}$. We have:

(i) If R is transitive, then it is $\aleph$-transitive, for all $\aleph \in \mathbb{N}$.

(ii) If R is $\aleph$-transitive, then it is e$\aleph$-transitive, for all $e \in \mathbb{N}$."

"**Definition 1.5.** Let (K, l) be a metric space and R_1, R_2 two binary relations on K. We say that (K, l) is (R_1, R_2)- regular if for sequence $\{ḩ_t\}$ in K such that $ḩ_t \to ḩ \in K$ as $t \to \infty$, and

$ḩ_t R_1 ḩ_{t+1}, \qquad ḩ_t R_2 ḩ_{t+1}, \qquad$ for all $t \in \mathbb{N}$,

there exists a subsequence $\{ḩ_{t(e)}\}$ such that

$ḩ_{t(e)} R_1 ḩ, \qquad ḩ_{t(e)} R_2 ḩ, \qquad$ for all $e \in \mathbb{N}$."

"**Definition 1.6.** We say that a subset D of K is (R_1, R_2) −directed if for all ḩ, ķ $\in D$, there exists $z \in K$ such that

$$ḩ R_1 z \text{ A } ķ R_1 z \text{ and } ḩ R_2 z \text{ A } ķ R_2 z."$$

"**Definition 1.7.** Let K be a set and $\alpha, \beta : K \times K \to [0, +\infty)$ are two mappings. Define two binary relations R_1 and R_2 on K by

ḩ, ķ $\in K$: ḩR_1ķ iff α(ḩ, ķ) ≤ 1

and

ḩ, ķ $\in K$: ḩR_2ķ iff β(ḩ, ķ) ≥ 1."

Berzig and Karapinar [4] introduced $(\alpha ƒ, \beta\varphi)$ −contractive mapping as given below:

"**Definition 1.8.** Let (K, l) be a metric space, and let $P : K \to K$ be a given mapping. We say that P is $(\alpha ƒ, \beta\phi)$ −contractive mappings if there exists a pair of generalized distance $(ƒ, \phi)$ such that

$ƒ(l(P$ḩ$, P$ķ$)) \leq \alpha($ḩ, ķ$)ƒ(l($ḩ, ķ$)) - \beta($ḩ, ķ$)\phi(l($ḩ, ķ$))$, for all ḩ, ķ $\in K$,

where $\alpha, \beta : K \times K \to [0, +\infty)$."

2. Main Results

Definition 2.1. Consider the pair of the functions $(ƒ, \varphi)$ and this pair of the functions is said to be strong generalized altering distance where $ƒ, \varphi : [0, \infty) \to [0, \infty)$ if the following conditions hold:

$(C1)\ \mathfrak{f}(0) = 0$,

$(C2)$ ƒ is continuous,

$(C3)$ ƒ is non-decreasing,

$(C4)\ \lim\limits_{t\to\infty} \varphi(x_t) = 0 \Longrightarrow \lim\limits_{t\to\infty} x_t = 0.$

Popescu in [16] and Moradi and Farajzadeh [15] introduced condition (C4).

Definition 2.2. Let (K, l) be a metric space, and let $P : K \to K$ be a given mapping. We say that P is generalized $(\alpha\mathfrak{f}, \beta\varphi)$ −contractive mappings if there exists a pair of generalized distance $(\mathfrak{f}, \varphi)$ such that

$$\mathfrak{f}(l(Ph, Pk)) \leq \alpha(h, k)\mathfrak{f}(Z(h, k)) - \beta(h, k)\varphi(Z(h, k)), \quad \text{for all } h, k \in K, \tag{2.1}$$

where $Z(h, k) = \max\{l(h, k), l(h, Ph), l(k, Pk), \frac{l(h, Ph)+l(k, Pk)}{2}\}$,

and $\alpha, \beta : K \times K \to [0, \infty)$.

Theorem 2.3. Let (K, l) be a complete metric space, $\mathbb{N}_0 \in \mathbb{N} \cup \{0\}$, and let $P : K \to K$ be generalized $(\alpha\mathfrak{f}, \beta\varphi)$ −contractive mapping satisfying the following conditions:

(i) R_i is N −transitive for $i = 1, 2$;

(ii) P is R_i −transitive for $i = 1, 2$;

(iii) There exists $h_0 \in K$ such that $h_0 R_i P h_0$ for $i = 1, 2$;

(iv) P is continuous.

Then P has a fixed point, that is, there exists $h \in K$ such that $Ph = h$.

Proof Let $h_0 \in K$ such that $h_0 R_i P h_0$ for $i = 1, 2$. Let sequence $\{h_t\}$ be defined by recursive relation $h_{t+1} = Ph_t$, for all $t \geq 0$.

If $h_t = h_{t+1}$, for some $t \geq 0$, then $h = h_t$ is a fixed point of P.

Assume that $h_t \neq h_{t+1}$, for all $t \geq 0$.

Form (ii) and (iii), we obtain

$$h_0 R_1 P h_0 \Rightarrow \alpha(h_0, Ph_0) = \alpha(h_0, h_1) \leq 1 \Longrightarrow \alpha(Ph_0, Ph_1) = \alpha(h_1, h_2) \leq 1.$$

Similarly, we have

$$h_0 R_2 P h_0 \Rightarrow \beta(h_0, Ph_0) = \beta(h_0, h_1) \geq 1 \Longrightarrow \beta(Ph_0, Ph_1) = \beta(h_1, h_2) \geq 1.$$

By Principal of Mathematical induction and using condition (ii),we have

$$\alpha(h_t, h_{t+1}) \leq 1, \quad \text{for all } t \geq 0. \tag{2.2}$$

and, similarly, we have

$\beta(h_t, h_{t+1}) \geq 1,$ for all $t \geq 0.$

(2.3)

Substituting $h = h_t$ and $k = h_{t+1}$ in (2.1), we obtain

$\mathfrak{f}(l(Ph_t, Ph_{t+1})) \leq \alpha(h_t, h_{t+1})\mathfrak{f}(Z(h_t, h_{t+1})) - \beta(h_t, h_{t+1})\varphi(Z(h_t, h_{t+1})),$

(2.4)

Using (2.2) and (2.3) in inequality (2.4), we obtain

$\mathfrak{f}(l(h_{t+1}, h_{t+2})) \leq \mathfrak{f}(Z(h_t, h_{t+1})) - \varphi(Z(h_t, h_{t+1})),$

(2.5)

where $Z(h_t, h_{t+1})$

$$= \max\{l(h_t, h_{t+1}), l(h_t, Ph_t), l(h_{t+1}, Ph_{t+1}), \frac{l(h_t, Ph_t) + l(h_{t+1}, Ph_{t+1})}{2}\}$$

$$= \max\{l(h_t, h_{t+1}), l(h_t, h_{t+1}), l(h_{t+1}, h_{t+2}), \frac{l(h_t, h_{t+1}) + l(h_{t+1}, h_{t+2})}{2}\}$$

$$= \max\{l(h_t, h_{t+1}), l(h_{t+1}, h_{t+2}), \frac{l(h_t, h_{t+1}) + l(h_{t+1}, h_{t+2})}{2}\}.$$

Case (i) If $Z(h_t, h_{t+1}) = l(h_t, h_{t+1})$.

From (2.5), we get

$$\mathfrak{f}(l(h_{t+1}, h_{t+2})) \leq \mathfrak{f}(l(h_t, h_{t+1})) - \varphi(l(h_t, h_{t+1})),$$

this implies,

$$\mathfrak{f}(l(h_{t+1}, h_{t+2})) \leq \mathfrak{f}(l(h_t, h_{t+1})).$$

Since f is non-decreasing function.

Therefore, $l(h_{t+1}, h_{t+2}) \leq l(h_t, h_{t+1})$.

Case (ii) If $Z(h_t, h_{t+1}) = l(h_{t+1}, h_{t+2})$.

From (2.5), we get

$$\mathfrak{f}(l(h_{t+1}, h_{t+2})) \leq \mathfrak{f}(l(h_{t+1}, h_{t+2})) - \varphi(l(h_{t+1}, h_{t+2})).$$

Since f is non-decreasing function.

Therefore, above inequality holds only when $l(h_{t+1}, h_{t+2}) = 0$,

this implies,

$h_{t+1} = h_{t+2}$, which is a contradiction.

Hence our supposition was wrong.

Therefore, $Z(h_t, h_{t+1}) \neq l(h_{t+1}, h_{t+2})$.

From above discussed cases, we get $l(h_{t+1}, h_{t+2}) \leq l(h_t, h_{t+1})$.

It shows that sequence $\{l(h_t, h_{t+1})\}$ is monotonically decreasing.

From above discussed cases, we also get $Z(h_t, h_{t+1}) = l(h_t, h_{t+1})$.
(2.6)

Using (2.6) in (2.5), we obtain

$\mathfrak{f}(l(h_{t+1}, h_{t+2})) \leq \mathfrak{f}(l(h_t, h_{t+1})) - \varphi(l(h_t, h_{t+1}))$,
(2.7)

Thus, there exists $r \in \mathbb{R}_+$ such that $\lim_{t\to\infty} l(h_t, h_{t+1}) = r$.
(2.8)

We will prove that $r = 0$.

Letting $t \to \infty$ in inequality (2.7), we get

$\lim_{t\to\infty} \mathfrak{f}(l(h_{t+1}, h_{t+2})) \leq \lim_{t\to\infty} \mathfrak{f}(l(h_t, h_{t+1})) - \lim_{t\to\infty} \varphi(l(h_t, h_{t+1}))$,

Using (2.8) in above inequality, we get

$\mathfrak{f}(r) \leq \mathfrak{f}(r) - \lim_{t\to\infty} \varphi(l(h_t, h_{t+1}))$,

this implies,

$$\lim_{t\to\infty} \varphi(l(h_t, h_{t+1})) = 0,$$

By using condition (C3), we get

$$\lim_{t\to\infty} l(h_t, h_{t+1}) = 0.$$

(2.9)

On the other hand, by (2.2) and (i), we obtain

$\alpha(h_u, h_{u+eN+1}) \leq 1$ for all $u, e \geq 0$.
(2.10)

and, similarly, we have

$\beta(h_u, h_{u+eN+1}) \geq 1$ for all $u, e \geq 0$.
(2.11)

Now, substituting $h = h_u$ and $k = h_{u'}$ in (2.1), where $u' = u + eN + 1$, for some $u, e \geq 0$, we obtain

$\mathfrak{f}(l(Ph_u, Ph_{u'})) \leq \alpha(h_u, h_{u'})\mathfrak{f}(Z(h_u, h_{u'})) - \beta(h_u, h_{u'})\varphi(Z(h_u, h_{u'}))$,
(2.12)

where $Z(h_u, h_{u'})$

$$= \max\left\{(h_u, h_{u'}), l(h_u, Ph_u), l(h_{u'}, Ph_{u'}), \frac{l(h_u, Ph_u) + l(h_{u'}, Ph_{u'})}{2}\right\}$$

$$= \max\left\{l(h_u, h_{u'}), l(h_u, h_{u+1}), l(h_{u'}, h_{u'+1}), \frac{l(h_u, h_{u+1}) + l(h_{u'}, h_{u'+1})}{2}\right\}.$$

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Using inequalities (2.10), (2.11) in (2.12), we get

$\mathfrak{f}(l(Ph_u, Ph_{u^F})) \leq \mathfrak{f}(Z(h_u, h_{u^F})) - \varphi(Z(h_u, h_{u^F})),$

(2.13)

where $Z(h_u, h_{u^F}) =$

$$\max\{l(h_u, h_{u^F}), l(h_u, h_{u+1}), l(h_{u^F}, h_{u^F+1}), \frac{l(h_u, h_{u+1}) + l(h_{u^F}, h_{u^F+1})}{2}\}.$$

Now we have three different subcases.

Subcase (i) If $Z(h_u, h_{u^F}) = l(h_u, h_{u^F})$.

Then inequality (2.13) becomes

$$\mathfrak{f}(l(h_{u+1}, h_{u^F+1})) \leq \mathfrak{f}(l(h_u, h_{u^F})) - \varphi(l(h_u, h_{u^F})).$$

Similarly, from case (i), we get $l(h_{u+1}, h_{u^F+1}) \leq l(h_u, h_{u^F})$.

It shows that sequence $\{l(h_u, h_{u^F})\}$ is monotonically decreasing.

Now repeating the same steps as after equation (2.8), we obtain

$$\lim_{u\to\infty} l(h_u, h_{u^F}) = 0.$$

Subcase (ii) If $Z(h_u, h_{u^F}) = l(h_u, h_{u+1})$.

Then inequality (2.13) becomes

$\mathfrak{f}(l(h_{u+1}, h_{u^F+1})) \leq \mathfrak{f}(l(h_u, h_{u+1})) - \varphi(l(h_u, h_{u+1})),$

Letting $u \to \infty$ and using (2.9), we get

$$\lim_{u\to\infty} \mathfrak{f}(l(h_{u+1}, h_{u^F+1})) = 0,$$

this implies,

$$\lim_{u\to\infty} l(h_{u+1}, h_{u^F+1}) = 0.$$

Subcase (iii) If $Z(h_u, h_{u^F}) = l(h_{u^F}, h_{u^F+1})$.

Then inequality (2.13) becomes

$$\mathfrak{f}(l(h_{u+1}, h_{u^F+1})) \leq \mathfrak{f}(l(h_{u^F}, h_{u^F+1})) - \varphi(l(h_{u^F}, h_{u^F+1})).$$

Similarly, from subcase (ii), we get

$$\lim_{u\to\infty} l(h_{u+1}, h_{u^F+1}) = 0.$$

From all above discussed subcases, we conclude that

$$\lim_{u\to\infty} l(h_u, h_{u^F}) = 0.$$

(2.14)

Next, we will prove that $\{h_t\}$ is Cauchy sequence. Suppose, to the contrary, that $\{h_t\}$ is not a Cauchy sequence.

Then there is $\in > 0$ and sequences $\{u(e)\}$ and $\{t(e)\}$ such that, for all positive integers e, we have

$$t(e) > u(e) > e, \qquad l(h_{u(e)}, h_{t(e)}) \geq \in \qquad \text{and} \qquad l(h_{u(e)}, h_{t(e)-1}) < \in$$

(2.15)

Then we have

$$\in \leq l(h_{u(e)}, h_{t(e)}) \leq l(h_{u(e)}, h_{t(e)-1}) + l(h_{t(e)-1}, h_{t(e)}) < \in + l(h_{t(e)-1}, h_{t(e)})$$

Letting $e \to \infty$ and using (2.9), we get

$$\lim_{e\to\infty} l(h_{u(e)}, h_{t(e)}) = \in. \tag{2.16}$$

Furthermore, for each $e \geq 0$, there exists $\mu_e, 5_e > 0$ such that $u'(e) = u(e) + N\mu_{e+1} + 1 = t(e) + 5_e$.

$$\in \leq l(h_{u(e)}, h_{u^F(e)}) \leq l(h_{u(e)}, h_{t(e)-1}) + \sum_{i=t(e)-1}^{u^F(e)-1} l(h_i, h_{i+1}) < \in + \sum_{i=t(e)-1}^{u^F(e)-1} l(h_i, h_{i+1})$$

Again, letting $e \to \infty$ and using (2.9), we get

$$\lim_{e\to\infty} l(h_{u(e)}, h_{u^F(e)}) = \in.$$

(2.17)

Again

$$l(h_{u(e)}, h_{u^F(e)}) \leq l(h_{u(e)}, h_{u(e)-1}) + l(h_{u(e)-1}, h_{u^F(e)-1}) + l(h_{u^F(e)-1}, h_{u^F(e)}),$$

$$l(h_{u(e)-1}, h_{u^F(e)-1}) \leq l(h_{u(e)-1}, h_{u(e)}) + l(h_{u(e)}, h_{u^F(e)}) + l(h_{u^F(e)}, h_{u^F(e)-1})$$

Letting $e \to \infty$ in above inequalities, using (2.9), (2.14) and (2.17), we get

$$\lim_{e\to\infty} l(h_{u(e)-1}, h_{u^F(e)-1}) = \in.$$

(2.18)

Substituting $h = h_{u(e)-1}$ and $k = h_{u^F(e)-1}$ in (2.1), we have

$$\mathfrak{f}(l(Ph_{u(e)-1}, Ph_{u^F(e)-1})) \leq \alpha(h_{u(e)-1}, h_{u^F(e)-1})\mathfrak{f}(Z(h_{u(e)-1}, h_{u^F(e)-1})) - \beta(h_{u(e)-1}, h_{u^F(e)-1})\varphi(Z(h_{u(e)-1}, h_{u^F(e)-1}))$$

Using (2.10) and (2.11) in above inequality, we get

$$\mathfrak{f}(l(h_{u(e)}, h_{u^F(e)})) \leq \mathfrak{f}(Z(h_{u(e)-1}, h_{u^F(e)-1})) - \varphi(Z(h_{u(e)-1}, h_{u^F(e)-1})),$$

(2.19)

where $Z(h_{u(e)-1}, h_{u^F(e)-1})$

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$$= \max\left\{ \begin{array}{c} l(h_{u(e)-1}, h_{u^F(e)-1}), l(h_{u(e)-1}, Ph_{u(e)-1}), l(h_{u^F(e)-1}, Ph_{u^F(e)-1}), \\ \frac{l(h_{u(e)-1}, Ph_{u(e)-1}) + l(h_{u^F(e)-1}, Ph_{u^F(e)-1})}{2} \end{array} \right\}$$

$$= \max\left\{ \begin{array}{c} l(h_{u(e)-1}, h_{u^F(e)-1}), l(h_{u(e)-1}, h_{u(e)}), l(h_{u^F(e)-1}, h_{u^F(e)}), \\ \frac{l(h_{u(e)-1}, h_{u(e)}) + l(h_{u^F(e)-1}, h_{u^F(e)})}{2} \end{array} \right\}.$$

Subsubcase (i) If $Z(h_{u(e)-1}, h_{u^F(e)-1}) = l(h_{u(e)-1}, h_{u^F(e)-1})$.

Then inequality (2.19) becomes

$$f\,(l(h_{u(e)}, h_{u^F(e)})) \le f\,(l(h_{u(e)-1}, h_{u^F(e)-1})) - \varphi\,(l(h_{u(e)-1}, h_{u^F(e)-1})),$$

Letting $e \to \infty$ in above inequality, using (2.17), (2.18) and the continuity of f and φ, we get

$$f(\in) \le f(\in) - \lim_{e \to \infty} \varphi\,(l(h_{u(e)-1}, h_{u^F(e)-1})).$$

Using the condition (C3), we conclude that $\in = 0$.

Subsubcase (ii) If $(h_{u(e)-1}, h_{u^F(e)-1}) = l(h_{u(e)-1}, h_{u(e)})$.

Then inequality (2.19) becomes

$$f\,(l(h_{u(e)}, h_{u^F(e)})) \le f\,(l(h_{u(e)-1}, h_{u(e)})) - \varphi\,(l(h_{u(e)-1}, h_{u(e)})),$$

Letting $e \to \infty$ in above inequality, using (2.9) and the continuity of f and φ, we get

$$f(\in) \le f(0) - \varphi(0) = 0,$$

this implies,

$$f(\in) = 0 \Longrightarrow \in = 0.$$

Subsubcase (iii) If $Z(h_{u(e)-1}, h_{u^F(e)-1}) = l(h_{u^F(e)-1}, h_{u^F(e)})$.

Then inequality (2.19) becomes

$$f\,(l(h_{u(e)}, h_{u^F(e)})) \le f\,(l(h_{u^F(e)-1}, h_{u^F(e)})) - \varphi\,(l(h_{u^F(e)-1}, h_{u^F(e)})).$$

Letting $e \to \infty$ in above inequality, using (2.9) and the continuity of f and φ, we get

$$f(\in) \le f(0) - \varphi(0) = 0,$$

this implies,

$$f(\in) = 0 \Longrightarrow \in = 0.$$

From all above discussed three subsubcases, we find $\in = 0$, which is a contraction with $\in >$ 0.

Hence our supposition was wrong. Hence, therefore $\{h_t\}$ is a Cauchy sequence.

Since (K, l) is a complete metric space, then there is $h \in K$ such that $\lim_{t \to \infty} h_t = h$.

Since P is continuous, then we have

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$$\mathfrak{h} = \lim_{t\to\infty} \mathfrak{h}_{t+1} = \lim_{t\to\infty} P\mathfrak{h}_t = P\mathfrak{h}.$$

Due to uniqueness of the limit, we derive that $P\mathfrak{h} = \mathfrak{h}$, that is, $\mathfrak{h}$ is a fixed point P.

Theorem 2.4 In Theorem 2.3, if we replace the continuity of P by the (R_1, R_2) −regularity of (K, l), then the conclusion of Theorem 2.3 holds.

Proof Following the proof of Theorem 2.3, we know that the sequence $\{\mathfrak{h}_t\}$ defined by $\mathfrak{h}_{t+1} = P\mathfrak{h}_t$ for all $t \geq 0$, converges to some $\mathfrak{h} \in K$. Since (K, l) is a complete metric space, then there exists $\mathfrak{h} \in K$ such that $\mathfrak{h}_t \to \mathfrak{h}$ as $t \to \infty$.

Furthermore, the sequence $\{\mathfrak{h}_t\}$ satisfies (2.2) and (2.3), that is,

$\mathfrak{h}_t R_1 \mathfrak{h}_{t+1}, \quad \mathfrak{h}_t R_2 \mathfrak{h}_{t+1}, \quad$ for all $t \in \mathbb{N}$.

Now, since is (R_1, R_2) −regular, then there exists a subsequence $\{\mathfrak{h}_{t(e)}\}$ of $\{\mathfrak{h}_t\}$ such that $\mathfrak{h}_{t(e)} R_1 \mathfrak{h}$, that is, $\alpha(\mathfrak{h}_{t(e)}, \mathfrak{h}) \leq 1$ and $\mathfrak{h}_{t(e)} R_2 \mathfrak{h}$, that is, $\beta(\mathfrak{h}_{t(e)}, \mathfrak{h}) \geq 1$, for all e. (2.20)

Substituting $\mathfrak{h} = \mathfrak{h}_{t(e)}$ and $\mathfrak{k} = \mathfrak{h}$, in (2.1), we obtain

$\mathfrak{f}(l(P\mathfrak{h}_{t(e)}, P\mathfrak{h})) \leq \alpha(\mathfrak{h}_{t(e)}, \mathfrak{h})\mathfrak{f}(Z(\mathfrak{h}_{t(e)}, \mathfrak{h})) - \beta(\mathfrak{h}_{t(e)}, \mathfrak{h})\varphi(Z(\mathfrak{h}_{t(e)}, \mathfrak{h}))$, for all e.

Using (2.20) in above inequality, we obtain

$\mathfrak{f}(l(P\mathfrak{h}_{t(e)}, P\mathfrak{h})) \leq \mathfrak{f}(Z(\mathfrak{h}_{t(e)}, \mathfrak{h})) - \varphi(Z(\mathfrak{h}_{t(e)}, \mathfrak{h}))$, for all e, (2.21)

where $Z(\mathfrak{h}_{t(e)}, \mathfrak{h}) = \max\{l(\mathfrak{h}_{t(e)}, \mathfrak{h}), l(\mathfrak{h}_{t(e)}, \mathfrak{h}_{t(e)+1}), l(\mathfrak{h}, P\mathfrak{h}), \frac{l(\mathfrak{h}_{t(e)}, P\mathfrak{h}_{t(e)}) + l(\mathfrak{h}, P\mathfrak{h})}{2}\}$

Case (i) If $Z(\mathfrak{h}_{t(e)}, \mathfrak{h}) = l(\mathfrak{h}_{t(e)}, \mathfrak{h})$.

Then inequality (2.21) becomes

$\mathfrak{f}(l(\mathfrak{h}_{t(e)+1}, P\mathfrak{h})) \leq \mathfrak{f}(l(\mathfrak{h}_{t(e)}, \mathfrak{h})) - \varphi(l(\mathfrak{h}_{t(e)}, \mathfrak{h}))$, for all e.

this implies,

$$\mathfrak{f}(l(\mathfrak{h}_{t(e)+1}, P\mathfrak{h})) \leq \mathfrak{f}(l(\mathfrak{h}_{t(e)}, \mathfrak{h})), \text{ for all e.}$$

Since $\mathfrak{f}$ is non-decreasing function. Therefore,

$$l(\mathfrak{h}_{t(e)+1}, P\mathfrak{h}) \leq l(\mathfrak{h}_{t(e)}, \mathfrak{h}), \text{ for all e.}$$

Letting e $\to \infty$ in above inequality, we obtain

$$l(\mathfrak{h}, P\mathfrak{h}) = 0 \Longrightarrow \mathfrak{h} = P\mathfrak{h}.$$

Case (ii) If $Z(\mathfrak{h}_{t(e)}, \mathfrak{h}) = l(\mathfrak{h}_{t(e)}, \mathfrak{h}_{t(e)+1})$.

Then inequality (2.21) becomes

$\mathfrak{f}(l(\mathfrak{h}_{t(e)+1}, P\mathfrak{h})) \leq \mathfrak{f}(l(\mathfrak{h}_{t(e)}, \mathfrak{h}_{t(e)+1})) - \varphi(l(\mathfrak{h}_{t(e)}, \mathfrak{h}_{t(e)+1}))$, for all e.

Letting $e \to \infty$ in above inequality, we obtain

$$l(h, Ph) = 0 \Longrightarrow h = Ph.$$

Case (iii) If $Z(h_{t(e)}, h) = l(h, Ph)$.

Then inequality (2.21) becomes

$\mathfrak{f}(l(h_{t(e)+1}, Ph)) \leq \mathfrak{f}(l(h, Ph)) - \varphi(l(h, Ph))$, for all e.

Letting $e \to \infty$ in above inequality, and the continuity of $\mathfrak{f}$ and φ, we get

$\mathfrak{f}(l(h, Ph)) \leq \mathfrak{f}(l(h, Ph)) - \varphi(l(h, Ph))$, for all e.

$$\lim_{e\to\infty} \varphi(l(h, Ph)) = 0.$$

By using condition (C3), we get

$$l(h, Ph) = 0 \Longrightarrow h = Ph.$$

From all above discussed three cases, we get $Ph = h$.

Theorem 2.5. Adding to the hypotheses of Theorem 2.3 (respectively, Theorem 2.4) that K is (R_1, R_2) −directed, we obtain uniqueness of the fixed point of P.

Proof Suppose that h and k are two fixed points of P. Since K is (R_1, R_2) −directed, there exists $z \in K$ such that

$\alpha(h, z) \leq 1, \alpha(k, z) \leq 1.$

(2.22)

and

$\beta(h, z) \geq 1, \beta(k, z) \geq 1.$

(2.23)

Since P is R_i −preserving for $i = 1, 2$, from (2.22) and (2.23), we get

$\alpha(h, P^t z) \leq 1, \alpha(k, P^t z) \leq 1,$ for all $t \geq 0.$

(2.24)

and

$\beta(h, P^t z) \geq 1, \beta(k, P^t z) \geq 1,$ for all $t \geq 0.$

(2.25)

Substituting $h = h, k = P^t z$ in (2.1), we have

$\mathfrak{f}(l(Ph, P(P^t z))) \leq \alpha(h, P^t z)\mathfrak{f}(Z(h, P^t z)) - \beta(h, P^t z)\varphi(Z(h, P^t z)).$

(2.26)

Using (2.24), (2.25) and (2.1), we obtain

$\mathfrak{f}(l(h, P^{t+1} z)) \leq \mathfrak{f}(Z(h, P^t z)) - \varphi(Z(h, P^t z)),$

(2.27)

where $Z(h, P^t z) = \max\{l(h, P^t z), l(h, Ph), l(P^t z, P^{t+1} z), \frac{l(h,Ph)+l(P^t Z,P^{t+1} Z)}{2}\}$

$$= \max\{l(h, P^t z), l(P^t z, P^{t+1} z), \frac{l(P^t Z,P^{t+1} Z)}{2}\}.$$

Case (i) If $Z(h, P^t z) = l(h, P^t z)$.

Then inequality (2.27) becomes

$\mathfrak{f}(l(h, P^{t+1} z)) \leq \mathfrak{f}(l(h, P^t z)) - \varphi(l(h, P^t z))$, for all $t \geq 0$.

(2.28)

Since f is non-decreasing function.

$l(h, P^{t+1} z) \leq l(h, P^t z)$, for all $t \geq 0$.

It follows that the sequence $\{l(h, P^{t+1} z)\}$ is decreasing. Thus there exists $r \geq 0$ such that

$$\lim_{t\to\infty} l(h, P^{t+1} z) = r.$$

We claim that $r = 0$.

Letting $t \to \infty$ in (2.28), we get

$$\mathfrak{f}(r) \leq \mathfrak{f}(r) - \lim_{t\to\infty} \varphi(l(h, P^t z)),$$

this implies,

$$\lim_{t\to\infty} \varphi(l(h, P^t z)) = 0.$$

(2.29)

By condition (C3), we obtain

$$\lim_{t\to\infty} l(h, P^t z) = 0.$$

(2.30)

Similarly, we get

$$\lim_{t\to\infty} l(k, P^t z) = 0.$$

Using (2.29) and (2.30), the uniqueness of the limit gives us $h = k$.

Case (ii) If $Z(h, P^t z) = l(P^t z, P^{t+1} z)$

Then inequality (2.27) becomes

$\mathfrak{f}(l(h, P^{t+1} z)) \leq \mathfrak{f}(l(P^t z, P^{t+1} z)) - \varphi(l(P^t z, P^{t+1} z))$, for all $t \geq 0$

(2.31)

Letting $t \to \infty$ in (2.31), we get

$$\lim_{t\to\infty} \mathfrak{f}(l(h, P^t z)) = 0,$$

this implies,

$$\lim_{t\to\infty} l(h, P^t z) = 0.$$

(2.32)

Similarly, we get

$$\lim_{t\to\infty} l(k, P^t z) = 0.$$

(2.33)

Using (2.32) and (2.33), the uniqueness of the limit gives us $h = k$.

Corollary 2.6. Let (K, l) be a complete metric space, and let $P : K \to K$ be a given mapping such that if there exists a pair of generalized distance $(\mathfrak{f}, \varphi)$ such that

$\mathfrak{f}(l(Ph, Pk)) \leq \alpha(h, k)\mathfrak{f}(l(h, k)) - \beta(h, k)\varphi(l(h, k))$, for all $h, k \in K$,

And $\alpha, \beta : K \times K \to [0, \infty)$.

Suppose

(i) R_i is N −transitive for $i = 1, 2$;

(ii) P is R_i −transitive for $i = 1, 2$;

(iii) There exists $h_0 \in K$ such that $h_0 R_i P h_0$ for $i = 1, 2$;

(iv) P is continuous.

Then P has a fixed point, that is, there exists $h \in K$ such that $Ph = h$.

Proof Taking $Z(h, k) = l(h, k)$ in Theorem 2.3 to get the proof.

Corollary 2.5. In Corollary 2.6, if we replace the continuity of P by the (R_1, R_2) −regularity of (K, l), then the conclusion of Corollary 2.6 holds.

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