A Parameter Uniform Numerical Method for a system of n Singularly Perturbed Robin type Initial Value Problems with Discontinuous Source Terms

S. Dinesh[1] **and M. Joseph Paramasivam**[2]

[1,2] *Department of Mathematics, Bishop Heber College (Affiliated to Bharathidasan University), Tiruchirappali-620017, Tamil Nadu, India.*
[1] *dineshselvaraj24@gmail.com*
[2] *paramasivam.ma@bhc.edu.in*

Abstract: *In this paper, a system of n singularly perturbed robin type initial value problems with discontinuous source terms is considered. The derivative component of each equation in the system is multiplied by a same singular perturbation parameter ε. A piecewise uniform Shishkin mesh is constructed and used, in conjunction with a classical finite difference scheme to form a numerical method for solving this problem. It is proved that the numerical approximations generated by this method are essentially first order convergent in the maximum norm at all points of the domain, uniformly with respect to the singular perturbation parameter. Numerical results are presented in support of the theory.*

Keywords: Singular pertubation problems, Robin initial conditions, Finite difference schemes, Discontinuous source terms, Shishkin mesh, Parameter uniform convergence.

1 Introduction

Consider a system of singularly perturbed robin type initial value problems with discontinuous source terms on the unit interval $\Omega = (0, 1]$, assume a single discontinuity in the source term at a point $d \in \Omega$. Let $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1]$ and the jump at d in any function is given as $[\omega](d) = \omega(d+) - \omega(d-)$. The corresponding initial value probles is to find $u_1, u_2, \ldots, u_n \in \mathbb{D} = \mathbb{C}^0(\overline{\Omega}) \cap \mathbb{C}^1(\Omega^- \cup \Omega^+)$, such that

$$\vec{L}\vec{u}(x) = E\vec{u}'(x) + A(x)\vec{u}(x) = \vec{f}(x), \quad x \in \Omega^- \cup \Omega^+ \tag{1}$$

with the prescribed initial conditions

$$\vec{\beta}\vec{u}(0) = \vec{u}(0) - \varepsilon\vec{u}'(0) = \vec{\phi} \tag{2}$$

where, $E = diag(\varepsilon, \varepsilon, \ldots, \varepsilon)$, $\vec{u}(x) = (u_1(x), u_2(x), \cdots, u_n(x))^T$, $A(x) = (a_{ij}(x))_{n\times n}$ and $\vec{f}(x) = (f_i(x))_{n\times 1}$.
The problem (1) and (2) can also be written in the operator form

$$\vec{L}\vec{u} = \vec{f} \text{ on } \Omega \tag{3}$$

with

$$\vec{\beta}\vec{u}(0) = \vec{\phi} \tag{4}$$

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where the operators $\vec{L}, \vec{\beta}$ are defined by

$$\vec{L} = ED + A, \quad \vec{\beta} = I - ED$$

where I is the identity operator, $D = \dfrac{d}{dx}$ is the first order differential operator.

Assumption 1 *The functions $a_{ij},\ f_i \in \mathbb{C}^{(2)}(\overline{\Omega}),\ i, j = 1(1)n$ satisfy the following positivity conditions*

$$\left.\begin{array}{ll} (i) & a_{ii}(x) > \sum\limits_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(x)| \ \ for\ i = 1(1)n \\ (ii) & a_{ij}(x) \leq 0 \quad for\ i \neq j\ \ and\ i = 1(1)n \end{array}\right\} \forall\, x \in \overline{\Omega}. \tag{5}$$

Assumption 2 *The positive number α satisfy the inequality*

$$0 < \alpha < \min_{\substack{i=1(1)n\\ x\in\overline{\Omega}}} \left\{ \sum_{j=1}^{n} a_{ij}(x) \right\}. \tag{6}$$

Assumption 3 *The singular perturbation parameters ε satisfy $0 < \varepsilon \leq 1$ is assumed to be distinct.*

The above problem is singularly perturbed in the following sense. The reduced problem obtained by putting $\varepsilon = 0$ in the system (1) is the linear algebraic system

$$A(x)\vec{v}(x) = \vec{f}(x), \quad x \in \Omega^- \cup \Omega^+ \tag{7}$$

where $A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ & & \vdots & \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix}$,

$\vec{v}(x) = (v_1(x), v_2(x), \cdots, v_n(x))^T$ and $\vec{f}(x) = (f_1(x), f_2(x), \cdots, f_n(x))^T$.

The source terms $f_1(x), f_2(x), \ldots, f_n(x)$ are sufficiently smooth on $\overline{\Omega}\backslash\{d\}$. The solution components $u_1, u_2, \ldots, u_n$ of the problem (1) and (2) have overlapping initial layers at $x = 0$ and have overlapping interior layers to the right side of point of discontinuity at $x = d$.

Theorem 1 *Let $A(x)$ satisfy* (5) *and* (6). *The problem* (1) - (2) *has a solution $\vec{u} \in \mathbb{D}$.*

Proof. The proof is by construction. Let $\vec{y}$ and $\vec{z}$ be the particular solutions of the differential equations

$$Ey_i'(x) + A(x)y_i(x) = f_i(x), \quad i = 1, 2, \ldots, n, \quad \text{for all } x \in \Omega^- \tag{8}$$

and

$$Ez_i'(x) + A(x)z_i(x) = f_i(x), \quad i = 1, 2, \ldots, n, \quad \text{for all } x \in \Omega^+ \tag{9}$$

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where $E = \begin{pmatrix} \varepsilon & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & \varepsilon \end{pmatrix}$, $A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix}$ respectively.

Consider the function

$$\vec{u}(x) = \begin{cases} y_i(x) + \vec{\beta}(u_i(0) - y_i(0))\phi_i(x), & i = 1, 2, \cdots, n, \ \ x \in \Omega^- \\ z_i(x) + B_i\phi_i(x), & i = 1, 2, \cdots, n, \ \ x \in \Omega^+ \end{cases} \tag{10}$$

where $\vec{\phi}$ is the solution of

$$\left.\begin{aligned} E\phi_i' + A(x)\phi_i(x) = \vec{0} \\ \beta_i\phi_i(0) = \vec{1}, \end{aligned}\right\} i = 1, 2, \cdots, n, \text{ for all } x \in \Omega.$$

Here $B_i, i = 1(1)n$ is chosen so that $\vec{u} \in \mathbb{D}$. In Ω, $0 < \vec{\phi} \leq 1$, there can be no internal maximum or minimum for $\vec{\phi}$ and hence $\phi_i' < 0, \ i = 1(1)n$ in Ω. Choose the constants B_i such that

$$\vec{y}(d-) = \vec{z}(d+)\vec{u}(d-) = \vec{u}(d+).$$

For the constants B_i to exist, it is required that

$$\frac{[u_i(0) - y_i(0)]\phi_i(d-)}{\phi(d+)} \neq 0 \qquad \text{for } i = 1(1)n.$$

Since $\phi_i(d+) > 0$ is true, the existence of $\vec{B}$ and hence $\vec{u}$ is ensured.

Remark: Throughout this paper, we use C as a generic positive constant vector which are independent of the perturbation parameters and the discretization parameter N.

2 Analytical Results

The operator $\vec{L}$ satisfies the following maximum principle.

Lemma 1 *Let $A(x)$ satisfy (5) and (6). Suppose that a function $\vec{u} \in \mathbb{D}$ satisfies $\vec{\beta}\vec{u}(0) \geq \vec{0}, \vec{L}\vec{u}(x) \geq \vec{0}$ for all $x \in \Omega^- \cup \Omega^+$. Then $\vec{u}(x) \geq \vec{0}$ for all $x \in \overline{\Omega}$.*

Proof. Let $u_i(p_i) = \min\limits_{x \in \overline{\Omega}}\{u_i(x)\}$, for $1 \leq i \leq n$. Without loss of generality assume that $u_1(p_1) \leq u_i(p_i)$, for $2 \leq i \leq n$. If $u_1(p_1) \geq 0$, then there is nothing to prove. Suppose that $u_1(p_1) < 0$, then the proof is by showing that this leads to contradiction. Note that $p_1 \neq \{0\}$, so either $p_1 \in \Omega^- \cup \Omega^+$ or $p_1 = d$.

Case (i): $p_1 \in \Omega^- \cup \Omega^+$,

$$\begin{aligned} \vec{\beta}\vec{u}(0) &= \vec{u}(0) - \varepsilon\vec{u}'(0) \\ &< 0, \quad \text{a contradiction} \end{aligned}$$

and

$$\begin{aligned} (\vec{L}\vec{u})_1(p_1) &= \varepsilon u_1'(p_1) + \sum_{j=1}^{n} a_{1j}(p_1)u_j(p_1) \\ &< 0, \quad \text{which is a contradiction.} \end{aligned}$$

Case (ii): $p_1 = d$,
since $\vec{u} \in \mathbb{C}(\Omega)$ and $u_1(d) < 0$, then there exists a neighbourhood $N_h = (d-h, d)$ such that $u_1(x) < 0$ for all $x \in N_h$. Now choose a point $x_1 \neq d$, $x_1 \in N_h$ such that $u_1(x_1) > u_1(d)$. It follows from the mean value theorem that, for some $x_2 \in N_h$, $u_1'(x_2) = \dfrac{u_1(d) - u_1(x_1)}{d - x_1} < 0$, since $x_2 \in N_h$.
Thus by similar argument of the first case, it follows that,

$$(\vec{L}\vec{u})_1(x_2) = \varepsilon u_1'(x_2) + \sum_{j=1}^{n} a_{1j}(x_2) u_j(x_2) < 0.$$

which is the contradiction.
As an immediate consequence of the above lemma the stability result is established in the following.

Lemma 2 *Let $A(x)$ satisfy* (5) *and* (6)*. Let $\vec{u}$ be the solution of* (1) *and* (2)*. Then,*

$$||\vec{u}(x)||_{\overline{\Omega}} \leq \max\left\{||\vec{\beta}\vec{u}(0)||, \frac{1}{\alpha}||\vec{L}\vec{u}||_{\Omega^- \cup \Omega^+}\right\}.$$

Proof. Define the two functions

$$\vec{\theta}^{\pm}(x) = \max\left\{\| \vec{\beta}\vec{u}(0) \|, \ \frac{1}{\alpha} \| \vec{L}\vec{u} \|_{\Omega^- \cup \Omega^+}\right\} \pm \vec{u}(x), \quad x \in \overline{\Omega}$$
$$\vec{\theta}^{\pm}(x) = M \pm \vec{u}(x)$$

where $M = \max\{||\vec{\beta}\vec{u}(0)||, \frac{1}{\alpha}||\vec{L}\vec{u}||_{\Omega^- \cup \Omega^+}\}$. Using the properties of $A(x)$, it is not hard to verify that $\vec{\beta}\vec{\theta}^{\pm}(0) \geq \vec{0}$ and $\vec{L}\vec{\theta}^{\pm}(x) \geq \vec{0}$ on $\Omega^- \cup \Omega^+$. It follows from Lemma 1 that $\vec{\theta}^{\pm}(x) \geq \vec{0}$ on $\overline{\Omega}$. Hence,

$$|\vec{u}(x)| \leq \max\left\{||\vec{\beta}\vec{u}(0)||, \frac{1}{\alpha}||\vec{L}\vec{u}||_{\Omega^- \cup \Omega^+}\right\}.$$

Lemma 3 *Let $A(x)$ satisfy* (5) *and* (6)*. Let $\vec{u}$ be the solution of* (1), (2)*. Then, for each i, $i = 1, 2, \cdots, n$ and $x \in \Omega^- \cup \Omega^+$, there exists a constant C such that*

$$|u_i(x)| \leq C\left\{\| \vec{\phi} \| + \| \vec{f} \|_{\Omega^- \cup \Omega^+}\right\}$$
$$|u_i'(x)| \leq C\varepsilon^{-1}\left\{\| \vec{\phi} \| + \| \vec{f} \|_{\Omega^- \cup \Omega^+}\right\}$$
$$|u_i''(x)| \leq C\varepsilon^{-2}\left\{\| \vec{\phi} \| + \| \vec{f} \|_{\Omega^- \cup \Omega^+} + \| \vec{f}' \|_{\Omega^- \cup \Omega^+}\right\}$$

Proof. From Lemma 2, it is evident that,

$$|\vec{u}(x)| \leq ||\vec{\beta}\vec{\psi}(0)|| + \frac{1}{\alpha}||\vec{L}\vec{\psi}||_{\Omega^- \cup \Omega^+}.$$

Thus,

$$|u_i(x)| \leq C\left\{\| \vec{\phi} \| + \| \vec{f} \|_{\Omega^- \cup \Omega^+}\right\}$$

Rewrite the differential equation (1), we get

$$\vec{u}'(x) = E^{-1}(\vec{f} - A\vec{u})$$
$$\text{Hence, } |u_i'(x)| \leq C\varepsilon^{-1}(||\vec{\phi}|| + ||\vec{f}||_{\Omega^- \cup \Omega^+})$$

Differentiating (1) once, we get

$$E\vec{u}''(x) + A(x)\vec{u}'(x) = f'(x) - A'(x)\vec{u}(x).$$

Using the bounds of $\vec{u}'$ and $\vec{u}$

$$|\vec{u}'(x)| \leq \varepsilon^{-1}[|\vec{f}'(x)| + C\varepsilon_i^{-1}(||\vec{\phi}|| + ||\vec{f}||) + C(||\vec{\phi}|| + ||\vec{f}||)]$$

and hence,

$$|u_i''(x)| \leq C\varepsilon^{-2}[||\vec{f}'|| + ||\vec{\phi}|| + ||\vec{f}||_{\Omega^- \cup \Omega^+}].$$

3 Estimates of derivatives

To derive sharper bounds on the derivatives of the solution, the solution is decomposed into a sum, composed of a regular component $\vec{v}$and a singular component $\vec{w}$. That is, $\vec{u} = \vec{v} + \vec{w}$. The regular component $\vec{v}$ is defined as the solution of the following problem:

$$\vec{L}\vec{v}(x) = \vec{f}(x), \quad x \in \Omega^- \cup \Omega^+$$
$$\vec{\beta}\vec{v}(0) = \vec{\beta}\vec{u}_0(0) \tag{11}$$

The singular component $\vec{w}$ is defined as the solution of the following problem

$$\vec{L}\vec{w}(x) = \vec{0}, \quad x \in \Omega^- \cup \Omega^+$$
$$\vec{\beta}\vec{w}(0) = \vec{\beta}(\vec{u} - \vec{v})(0), \quad [\vec{w}](d) = -[\vec{v}](d). \tag{12}$$

Theorem 2 *Let $A(x)$ satisfy (5) and (6). Then the components v_i, $i = 1(1)n$ of the regular component $\vec{v}$ and its derivatives satisfy the bounds for all $x \in \Omega^- \cup \Omega^+$ and $k = 0, 1, 2$,*

$$||\vec{v}^{(k)}||_{\Omega^- \cup \Omega^+} \leq C \;\; for \;\; k = 0, 1$$
$$|[\vec{v}](d)| \leq C, \quad |[\vec{v}'](d)| \leq C$$
$$||v_i''||_{\Omega^- \cup \Omega^+} \leq C\varepsilon^{-1} \;\; for\, i = 1(1)n.$$

Proof. Following the techniques in [], one can arrive at the results

$$||\vec{v}^{(k)}||_{\Omega^- \cup \Omega^+} \leq C \;\; \text{for} \;\; k = 0, 1$$

Also for $i = 1, 2, \cdots, n$,

$$||v_i''||_{\Omega^- \cup \Omega^+} \leq C\varepsilon^{-1}$$

and

$$|[v_i](d)| = v_i(d+) - v_i(d-) \leq |v_i(d+)| + |v_i(d-)| \leq C.$$

Similarly, $|[\vec{v}'](d)| \leq C$, and hence the proof is completed. Now bounds on the layer components of $\vec{u}$ are to be found. Consider the layer functions

$$B_{l_i}(x) = e^{-\alpha x/\varepsilon}, \quad B_{r_i}(x) = e^{-\alpha(x-d)/\varepsilon}, \quad i = 1(1)n.$$

Theorem 3 *Let $A(x)$ satisfy (5) and (6). Then the components w_i, $i = 1(1)n$ of the regular component $\vec{w}$ and its derivatives satisfy the bounds for all $x \in \Omega^- \cup \Omega^+$*

$$|w_i(x)| \leq \begin{cases} CB_{l_n}(x), & x \in \Omega^- \\ CB_{r_n}(x), & x \in \Omega^+ \end{cases}$$

$$|w_i'(x)| \leq \begin{cases} C\varepsilon^{-1} \sum_{q=i}^{n} B_{l_q}(x), & x \in \Omega^- \\ C\varepsilon^{-1} \sum_{q=i}^{n} B_{r_q}(x), & x \in \Omega^+ \end{cases}$$

$$|w_i''(x)| \leq \begin{cases} C\varepsilon^{-1} \sum_{q=1}^{n} B_{l_q}(x), & x \in \Omega^- \\ C\varepsilon^{-1} \sum_{q=1}^{n} B_{r_q}(x), & x \in \Omega^+ \end{cases}$$

Proof. We have $\vec{u} = \vec{v} + \vec{w}$ and by Lemma 2 $|\vec{w}(0)| \leq C$ and $|\vec{w}(d+)| \leq C$. Define the barrier function

$$\xi = CB_{l_n}(x)\vec{e}$$

with C chosen sufficiently large such that $\xi \geq |\vec{w}|$ at $x = 0, d+$,

$$\begin{aligned} \vec{L}\xi &= CB_{l_n} \left(\sum_{j=1}^{n} a_{1j} - \alpha, \sum_{j=1}^{n} a_{2j} - \alpha, \cdots, \sum_{j=1}^{n} a_{nj} - \alpha \right) \\ &\geq \vec{0} = |\vec{L}\vec{w}|_{\Omega^-} \end{aligned}$$

and it is not hard to see that $\vec{\beta}\xi(0) \geq \vec{0}$. Using maximum principle (1), we get the required bounds on $\vec{w}$. Now to bound first-order derivative of w_i, consider $\varepsilon w_i' + \sum_{j=1}^{n} a_{ij} w_j = 0$, together with the bound on $\vec{w}$. This implies that

$$|w_i'(x)| \leq \begin{cases} C\varepsilon^{-1} B_{l_n}(x), & x \in \Omega^- \\ C\varepsilon^{-1} B_{r_n}(x), & x \in \Omega^+ \end{cases}$$

Now to find the sharper bound consider the system of $n-1$ equations

$$\vec{\tilde{E}}\vec{\tilde{w}}' + \vec{\tilde{A}}\vec{\tilde{w}} = \vec{h},$$

where $\tilde{E}, \tilde{A}$ are the matrix obtained by deleting the last row and column from E, A respectively and the components of $\vec{h}$ are $h_i = -a_{in} w_n$, for $1 \leq i \leq n-1$. Using the bounds derived earlier and the decomposition of $\vec{\tilde{w}} = \vec{q} + \vec{r}$, into regular and singular component we get the required result. Now to bound second-order derivatives, differentiate $\varepsilon w_i' + \sum_{j=1}^{n} a_{ij} w_j = 0$ once and using the estimates of w_i', we get the required bounds on singular component $\vec{w}$ and its derivatives.

Lemma 4 *For all i, j such that $1 \leq i \leq j \leq n$, there exists a unique point $x_{i,j} \in (0, d)$ such that $\varepsilon^{-1} B_{l_i}(x_{i,j}) = \varepsilon^{-1} B_{l_j}(x_{i,j})$. Also, $\varepsilon^{-1} B_{r_i}(d + x_{i,j}) = \varepsilon^{-1} B_{r_j}(d + x_{i,j})$. On $[0, x_{i,j})$ we have $\varepsilon^{-1} B_{l_i}(x) > \varepsilon^{-1} B_{l_j}(x)$ and on $(x_{i,j}, d)$ we have $\varepsilon^{-1} B_{l_i}(x) < \varepsilon^{-1} B_{l_j}(x)$. Similarly, on $(d, d + x_{i,j})$ we have $\varepsilon^{-1} B_{r_i}(x) > \varepsilon^{-1} B_{r_j}(x)$ and on $(d + x_{i,j}, 1]$ we have $\varepsilon^{-1} B_{r_i}(x) < \varepsilon^{-1} B_{r_j}(x)$.*

For the analysis of the convergence, a more precise decomposition of the components of the singular component $\vec{w}$ is required. The next Lemma provides the necessary estimates of decomposed layer functions.

Theorem 4 *The singular component $\vec{w}$ can be decomposed in this way as follows, for* $1 \leq i \leq n$:

$$w_i(x) = \sum_{q=1}^{n} w_{i,q}(x)$$

where

$$|w'_{i,q}(x)| \leq \begin{cases} C\varepsilon^{-1}B_{l_q}(x), & x \in \Omega^- \\ C\varepsilon^{-1}B_{r_q}(x), & x \in \Omega^+ \end{cases} \quad |w''_{i,q}(x)| \leq \begin{cases} C\varepsilon^{-1}B_{l_q}(x), & x \in \Omega^- \\ C\varepsilon^{-1}B_{r_q}(x), & x \in \Omega^+ \end{cases}$$

Proof. Define a function $w_{i,1}$ as follows

$$w_{i,1}(x) = w_i(x) - \sum_{q=2}^{n} w_{i,q}(x)$$

and for $1 < q \leq n$, we have

$$w_{i,q} = \begin{cases} \sum\limits_{k=0}^{2} \frac{[(x-x_{q-1,q})^k]}{k!} w_i^{(k)}(x_{q-1,q}), & x \in [0, x_{q-1,q}), \\ w_i(x) - \sum\limits_{r=q+1}^{n} w_{i,r}(x), & x \in [x_{q-1,q}, d), \\ \sum\limits_{k=0}^{2} \frac{[(x-d-x_{q-1,q})^k]}{k!} w_i^{(k)}(d + x_{q-1,q}), & x \in (d, d + x_{q-1,q}), \\ w_i(x) - \sum\limits_{r=q+1}^{n} w_{i,r}(x), & x \in [d + x_{q-1,q}, 1] \end{cases}$$

Now we establish the bounds on the second derivative.
For $x \in [x_{n-1,n}, d] \cup [d + x_{n-1,n}, 1]$,

$$|\varepsilon w''_{i,n}(x)| = |\varepsilon w''_i(x)| \leq C\varepsilon^{-1} \sum_{q=1}^{n} B_{l_q}(x) \leq C\varepsilon^{-1} B_{l_n}(x).$$

For $x \in [0, x_{n-1,n}) \cup (d, d + x_{n-1,n})$,

$$|\varepsilon w''_{i,n}(x)| = |\varepsilon w''_i(x_{n-1,n})| \leq C\varepsilon^{-1} \sum_{q=1}^{n} B_{l_q}(x_{n-1,n}) \leq C\varepsilon^{-1} B_{l_n}(x_{n-1,n}) \leq C\varepsilon^{-1} B_{l_n}(x).$$

Now for each $2 \geq q \geq n-1$, it follows that
For $x \in [x_{q-1,q}, d) \cup [d + x_{q-1,q}, 1]$,

$$w''_{i,q}(x) = 0.$$

For $x \in [0, x_{q-1,q}) \cup (d, d + x_{q-1,q}]$,

$$|\varepsilon w''_{i,q}(x)| = |\varepsilon w''_i(x_{q-1,q})| \leq C\varepsilon^{-1} \sum_{q=1}^{n} B_{l_q}(x_{q-1,q}) \leq C\varepsilon^{-1} B_{l_q}(x_{q-1,q}) \leq C\varepsilon^{-1} B_{l_q}(x).$$

For $x \in [x_{1,2}, d) \cup [d + x_{1,2}, 1]$,

$$w''_{i,1}(x) = 0.$$

For $x \in [0, x_{1,2}) \cup (d, d + x_{1,2}]$,

$$|\varepsilon w''_{i,1}(x)| = |\varepsilon w''_i(x) - \sum_{q=2}^{n} \varepsilon w''_{i,q}(x)| \leq C\varepsilon^{-1} \sum_{q=1}^{n} B_{l_q}(x) \leq C\varepsilon^{-1} B_{l_1}(x).$$

For the bounds on the first derivatives we have the relation

$$|w'_{i,q}(x)| = \left| \int_x^{x_{q,q+1}} w''_{i,q}(t)dt \right| \leq C\varepsilon^{-1} \left| \int_x^{x_{q,q+1}} B_{l_q}(t)dt \right| \leq C\varepsilon^{-1} B_{l_q}(x).$$

4 The Shishkin mesh

A piecewise uniform mesh with N mesh-intervals is constructed and mesh points $\{x_j\}_{j=0}^N$ are obtained by dividing the interval $\overline{\Omega}$ into $2n + 2$ sub-intervals as follows.

$$\overline{\Omega} = [0, \sigma_1] \cup (\sigma_1, \sigma_2] \cdots (\sigma_{n-1}, \sigma_n] \cup (\sigma_n, d] \cup (d, d + \tau_1] \cup (d + \tau_1, d + \tau_2] \cup \cdots (d + \tau_{n-1}, d + \tau_n] \cup (d + \tau_n, 1].$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n,\ \tau_1, \tau_2, \ldots, \tau_n$ are the transition parameters satisfying

$$0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n \leq \frac{d}{2} \quad \text{and} \quad d < \tau_1 < \tau_2 < \cdots < \tau_n \leq \frac{1-d}{2}.$$

The interior points of the mesh are denoted by

$$\Omega^N = \left\{ x_i : 1 \leq i \leq \frac{N}{2} - 1 \right\} \cup \left\{ x_i : \frac{N}{2} + 1 \leq i \leq N - 1 \right\} = \Omega^{-N} \cup \Omega^{+N}$$

Let $h_i = x_i - x_{i-1}$ be the i^{th} mesh step and $h_i = \dfrac{h_i + h_{i+1}}{2}$, clearly $x_{\frac{N}{2}} = d$. Then on the sub-intervals $[0, \sigma_1]$ and $[d, d + \tau_1]$ a uniform mesh with $\dfrac{N}{2^{2n}}$ mesh intervals are placed and similarly on $(\sigma_k, \sigma_{k+1}], (d + \tau_k, d + \tau_{k+1}],\ 1 \leq k \leq n - 1$, a uniform mesh with $\dfrac{N}{2^{2n-2k+2}}$ mesh intervals and on $(\sigma_n, d]$ and $(d + \tau_n, 1]$ a uniform mesh of $\dfrac{N}{4}$ mesh intervals are placed.
The $2n$ transition points between the uniform meshes are defined by

$$\sigma_n = \min \left\{ \frac{d}{2}, \frac{\varepsilon}{\alpha} \ln N \right\}, \qquad \tau_n = \min \left\{ \frac{1-d}{2}, \frac{\varepsilon}{\alpha} \ln N \right\}$$

and for $r = n - 1, \ldots, 2, 1$,

$$\sigma_r = \min \left\{ \frac{\sigma_{r+1}}{2}, \frac{\varepsilon}{\alpha} \ln N \right\}, \qquad \tau_r = \min \left\{ \frac{\tau_{r+1}}{2}, \frac{\varepsilon}{\alpha} \ln N \right\}$$

This construction leads to a class of 2^{2n} piecewise uniform Shishkin meshes.

5 The Discrete Problem

The Initial Value Problem (1), (2) is discretised using a fitted mesh method composed of a classical finite difference operator on a piecewise uniform fitted mesh $\overline{\Omega}^N$. Then the fitted mesh method for solving the system (1) and (2) is, for $i = 1, 2, \ldots, n$,

$$(\vec{L}^N\vec{U})_i(x_j) = ED^-\vec{U}(x_j) + A(x_j)\vec{U}(x_j) = \vec{f}(x_j), \quad j \neq \frac{N}{2} \tag{13}$$

with

$$\vec{\beta}\vec{U}(0) = \vec{U}(0) - \varepsilon D^+\vec{U}(0) = \vec{\phi} \tag{14}$$

and at $x_{\frac{N}{2}} = d$, the scheme is given by

$$\vec{L}^N\vec{U}(x_{\frac{N}{2}}) = ED^-\vec{U}(x_{\frac{N}{2}}) + A(x_{\frac{N}{2}})\vec{U}(x_{\frac{N}{2}}) = \vec{f}(x_{\frac{N}{2}} - 1).$$

The problem (13), (14) can also be written in the operator form

$$L^N\vec{U} = \vec{f} \text{ on } \Omega^N \text{ with}$$

$$\vec{\beta}^N\vec{U}(0) = \vec{\phi}$$

$$\text{where } L^N = ED^- + A \ \text{ with}$$

$$\vec{\beta}^N = I - \varepsilon D^+ I$$

and D^+, D^- are the difference operators

$$D^-\vec{U}(x_j) = \frac{\vec{U}(x_j) - \vec{U}(x_{j-1})}{x_j - x_{j-1}}, \quad D^+\vec{U}(x_j) = \frac{\vec{U}(x_{j+1}) - \vec{U}(x_j)}{x_{j+1} - x_j}, \ \ j = 1, 2, \ldots, N.$$

The following discrete results are analogous to those for the continuous case.

Lemma 5 *Let $A(x)$ satisfy (5) and (6). Suppose that a mesh function $\vec{Z}(x_j)$ satisfies $\vec{\beta}\vec{Z}(x_0) \geq \vec{0}$ and $\vec{L}^N\vec{Z}(x_j) \geq \vec{0}$, for all $x_j \in \Omega^N$ and $(D^+ - D^-)\vec{Z}(x_{\frac{N}{2}}) \leq \vec{0}$, implies that $\vec{Z}(x_j) \geq \vec{0}$ for all $x_j \in \overline{\Omega}$.*

Proof. Let x_q be any point at which $\vec{Z}(x_q)$ attains its minimum on $\overline{\Omega}^N$. If $\vec{Z}(x_q) \geq \vec{0}$, then there is nothing to prove. Without loss of generality, Suppose that $Z_1(x_q) < 0$, then clearly, $x_q \neq 0$. If $x_q = 0$, then

$$\begin{aligned}\vec{\beta}^N\vec{Z}(0) &= \vec{Z}(0) - \varepsilon D^+\vec{Z}(0)\\ &< 0, \quad \text{a contradiction.}\end{aligned}$$

Therefore, $x_q \neq 0$. If $q \neq N/2$, it is clear that

$$D^-Z_1(x_q) \leq 0 \leq D^+Z_1(x_q)$$

and hence if $x_q \in \Omega^N$, $q \neq N/2$, then

$$(\vec{L}^N\vec{Z})_1(x_q) = \varepsilon D^-Z(x_q) + a_{11}(x_q)Z_1(x_q) + \cdots + a_{1n}(x_q)Z_n(x_q) < 0$$

which is a contradiction. Hence, the only possiblity is that $x_q = x_{\frac{N}{2}}$. Then

$$D^- Z_1(x_{\frac{N}{2}}) \leq 0 \leq D^+ Z_1(x_{\frac{N}{2}}) \leq D^- Z_1(x_{\frac{N}{2}}).$$

From the above it is observed that

$$Z_1(x_{\frac{N}{2}-1}) = Z_1(x_{\frac{N}{2}}) = Z_1(x_{\frac{N}{2}+1}) < 0$$

then, $(\vec{L}^N \vec{Z})_1(x_{\frac{N}{2}-1}) < 0$, which is a cotradiction. Hence the result.

Lemma 6 *Let $A(x)$ satisfy* (5) *and* (6)*. If $\vec{U}$ be the numerical solution of* (1) *and* (2)*, then*

$$||\vec{U}(x_j)||_{\overline{\Omega}^N} \leq \max\left\{||\vec{\beta}\vec{U}(0)||, \frac{1}{\alpha}||\vec{f}||_{\Omega^{-N}\cup\Omega^{+N}}\right\}.$$

Proof. Define the two mesh functions

$$\vec{\Theta}^{\pm}(x_j) = \max\left\{||\vec{\beta}^N\vec{\Psi}(0)||, \frac{1}{\alpha}||\vec{f}||_{\Omega^{-N}\cup\Omega^{+N}}\right\} \pm \vec{U}(x_j).$$

Using the properties of $A(x)$, it is not hard to verify that $\vec{\beta}^N\vec{\Theta}^{\pm}(0) \geq \vec{0}$ and $L^N\vec{\Theta}^{\pm} \geq \vec{0}$ on Ω^N. Applying the discrete maximum principle (Lemma 5) then gives $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\overline{\Omega}^N$, and so

$$|\vec{U}(x_j)| \leq \max\left\{||\vec{\beta}^N\vec{\Psi}(0)||, \frac{1}{\alpha}||\vec{f}||_{\Omega^{-N}\cup\Omega^{+N}}\right\}$$

as required.

6 The Local Truncation Error

From Lemma 6, it is seen that in order to bound the error $||\vec{U} - \vec{u}||$, it suffices to bound $\vec{L}^N(\vec{U} - \vec{u})$. Notice that, for $x_j \in \Omega^N$,

$$\begin{aligned}\vec{L}^N(\vec{U}(x_j) - \vec{u}(x_j)) &= \vec{L}^N\vec{U}(x_j) - \vec{L}^N\vec{u}(x_j)\\ &= E(D^- - D)\vec{u}(x_j)\end{aligned}$$

and

$$((\vec{L} - \vec{L}^N)u)_i(x_j) = \varepsilon(D^- - D)v_i(x_j) + \varepsilon(D^- - D)w_i(x_j)$$

which is the local truncation of the first derivative. Then, by the triangle inequality,

$$|(\vec{L}^N(\vec{U} - \vec{u}))_i(x_j)| \leq |\varepsilon(D^- - D)v_i(x_j)| + |\varepsilon(D^- - D)w_i(x_j)|.$$

Analogous to the continuous case, the discrete solution $\vec{U}$ can be decomposed into $\vec{V}$ and $\vec{W}$ which are defined to be solutions of the following discrete problems

$$(\vec{L}^N\vec{V})(x_j) = \vec{f}(x_j) \text{ on } \Omega^N, \quad \vec{\beta}^N\vec{V}(0) = \vec{\beta}\vec{v}(0) \tag{15}$$

and

$$(\vec{L}^N\vec{W})(x_j) = \vec{0} \text{ on } \Omega^N, \quad \vec{\beta}^N\vec{W}(0) = \vec{\beta}\vec{w}(0) \tag{16}$$

where $\vec{v}$ and $\vec{w}$ are the solutions of (11) and (12) respectively.
Further, for $i = 1, 2, \ldots, n$,

$$|(\vec{\beta}^N(\vec{V} - \vec{v}))_i(0)| = |(D - D^+)v_i(0)|$$
$$|(\vec{\beta}^N(\vec{W} - \vec{w}))_i(0)| = |(D - D^+)w_i(0)|$$

$$|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| = |\varepsilon(D^- - D)v_i(x_j)| \tag{17}$$

$$|(\vec{L}^N(\vec{W} - \vec{w}))_i(x_j)| = |\varepsilon(D^- - D)w_i(x_j)|. \tag{18}$$

The error at each point $x_j \in \overline{\Omega}^N$ is denoted by $\vec{U}(x_j) - \vec{u}(x_j)$. Then the local truncation error $\vec{L}^N(\vec{U}(x_j) - \vec{u}(x_j))$ has the decomposition

$$\vec{L}^N(\vec{U} - \vec{u})(x_j) = \vec{L}^N(\vec{V} - \vec{v})(x_j) + \vec{L}^N(\vec{W} - \vec{w})(x_j).$$

By a Taylor expansion on regular and singular components, we have

$$|\varepsilon\left(\frac{d}{dx} - D^-\right)v_k(x_j)| \le C\varepsilon\frac{(x_j - x_{j-1})}{2}|v_k|_2 \le CN^{-1} \tag{19}$$

and

$$|\varepsilon\left(\frac{d}{dx} - D^-\right)w_k(x_j)| \le \begin{cases} C\varepsilon\frac{(x_j - x_{j-1})}{2}|w_k|_2 \\ C\varepsilon \max\limits_{[x_j, x_{j-1}]}|w_k'| \end{cases} \tag{20}$$

where $k = 1, 2, \ldots, n, \;\; j \ne \frac{N}{2}$.
The error in the smooth and singular components are bounded in the following section.

7 Error Analysis

The proof of the theorem on the error estimate is split into two parts. First, a theorem concerning the error in the smooth component is established. Then the error in the singular component is established.

Theorem 5 *Let $A(x)$ satisfy (5) and (6). Let $\vec{v}$ denote the smooth component of the solution of (1), (2) and $\vec{V}$ denote the smooth component of the solution of the problem (13), (14). Then*

$$|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| \le CN^{-1}$$

Proof. From the expression (19),

$$\begin{aligned} |(\vec{\beta}^N(\vec{V} - \vec{v})_i(0)| &\le C(x_1 - x_0) \max_{s\in[x_0,x_1]}|v_i''(s)| \\ &\le CN^{-1} \end{aligned} \tag{21}$$

It is not hard to find that

$$\begin{aligned} \left|\varepsilon(D^- - D)v_i(x_j)\right| &\le Ch_j \max_{s\in I_j}|\varepsilon v_i''(s)| \\ &\le Ch_j \\ &\le CN^{-1} \end{aligned}$$

as required.

Lemma 7 *Let $A(x)$ satisfy* (5) *and* (6). *Let $\vec{w}$ denote the smooth component of the solution of* (1), (2) *and $\vec{W}$ denote the smooth component of the solution of the problem* (13), (14). *Then*

$$|(\vec{L}^N(\vec{W}-\vec{w}))_i(x_j)| \leq CN^{-1}\ln N$$

Proof. For the proof of this theorem, we have to evaluate the error estimates for the singular components on different subintervals considered as follows:
Case (i): For $x_j \in [\sigma_n, d) \cup [d+\tau_n, 1]$.

From the expression (20),

$$\begin{aligned}|(\vec{\beta}^N(\vec{W}-\vec{w}))_i(0)| &\leq C\varepsilon(x_1-x_0)\max_{[x_0,x_1]}|w_i''| \\ &\leq CN^{-1}\ln N\end{aligned}$$

Using (20) and bounds on singular components, we have for $i=1,2,\ldots,n$

$$\begin{aligned}|((\vec{L}^N-\vec{L})\vec{w})_i(x_j)| &\leq C\varepsilon\sum_{q=i}^{n}\frac{B_{l_q}(x)}{\varepsilon_q} \\ &\leq C||B_{l_n}||_{[x_{i-1},x_i]} = B_{l_n}(x_{i-1}) \\ &\leq CN^{-1}.\end{aligned}$$

Similar arguments prove a similar result for the subinterval $[d+\tau_n, 1]$. Hence, for $x_j \in [\sigma_n, d) \cup [d+\tau_n, 1]$ we have

$$|((\vec{L}^N-\vec{L})\vec{w})_i(x_j)| \leq CN^{-1}.$$

Case (ii): For $x_j \in (0, \sigma_1] \cup (d, d+\tau_1]$.

Using (20) and bounds on singular components yields

$$\begin{aligned}|((\vec{L}^N-\vec{L})\vec{w})_i(x_j)| &\leq C(x_i-x_{i-1})||\varepsilon w_i''|| \\ &\leq h_i\varepsilon^{-1}\sum_{q=1}^{n}B_{l_q}(x) \\ &\leq CN^{-1}\ln N.\end{aligned}$$

Case (iii): For $x_j \in (\sigma_r, \sigma_{r+1}) \cup (d+\tau_r, d+\tau_{r+1})$, where $1 \leq r \leq n-1$.

Using the decomposition in Theorem 4 of singular components and bounds on singular components gives

$$|((\vec{L}^N-\vec{L})\vec{w})_i(x_j)| = \left|\sum_{q=1}^{n-1}\varepsilon\left(\frac{d}{dx}-D^-\right)w_{i,q}(x_j) + \varepsilon\left(\frac{d}{dx}-D^-\right)w_{i,n}(x_j)\right|. \qquad (22)$$

Consider the first part of (22) and using the bounds on singular components, we obtain

$$\begin{aligned}\left|\sum_{q=1}^{n-1}\varepsilon\left(\frac{d}{dx}-D^-\right)w_{i,q}(x_j)\right| &\leq ||\sum_{q=1}^{n-1}\varepsilon w_{i,q}''||_{[x_{i-1},x_i]} \\ &\leq CB_{l_{n-1}}(x_{i-1}) \\ &\leq CN^{-1}.\end{aligned}$$

Using the bounds on singular components for the second part of (22), we have

$$\begin{aligned}\left|\varepsilon\left(\frac{d}{dx}-D^{-}\right)w_{i,n}(x_j)\right| &\leq \frac{h_i}{2}||\varepsilon w''_{i,n}|| \\ &\leq CN^{-1}\ln N.\end{aligned}$$

Case (iv): For $x_j \in \{\sigma_r, d+\tau_r\}$, where $1 \leq r \leq n-1$.

Using the decomposition of the singular components and bounds on singular components defined in Theorem 4 gives

$$|((\vec{L}^N-\vec{L})\vec{w})_i(x_j)| = \left|\sum_{q=1}^{n-1}\varepsilon\left(\frac{d}{dx}-D^{-}\right)w_{i,q}(x_j)+\varepsilon\left(\frac{d}{dx}-D^{-}\right)w_{i,n}(x_j)\right|. \quad (23)$$

Consider the first part of (23) for the case $i \leq r$, and using the bounds on singular components, we obtain

$$\begin{aligned}\left|\sum_{q=1}^{n-1}\varepsilon\left(\frac{d}{dx}-D^{-}\right)w_{i,q}(x_j)\right| &\leq ||\sum_{q=1}^{n-1}\varepsilon w'_{i,q}||_{[x_{i-1},x_i]} \\ &\leq CN^{-1}.\end{aligned}$$

and if $i > r$, using the bounds on singular components and the analysis in Case (i), we have

$$\begin{aligned}\left|\sum_{q=1}^{n-1}\varepsilon\left(\frac{d}{dx}-D^{-}\right)w_{i,q}(x_j)\right| &\leq ||\sum_{q=1}^{n-1}\varepsilon w'_{i,q}||_{[x_{i-1},x_i]} \\ &\leq CN^{-1}.\end{aligned}$$

For the second part of (23), use bounds on singular components defined in Theorem 4, to obtain

$$\begin{aligned}\left|\varepsilon\left(\frac{d}{dx}-D^{-}\right)w_{i,n}(x_j)\right| &\leq Ch_r||\varepsilon w''_{i,n}|| \\ &\leq CN^{-1}\ln N.\end{aligned}$$

Now at the point $x_{\frac{N}{2}} = d$,

$$\begin{aligned}|(\vec{L}^N(\vec{U}-\vec{u}))_i(d)| &\leq C\varepsilon h^{+}\max_{[x_{\frac{N}{2}},x_{\frac{N}{2}+1}]}|u''_i(\eta)| + C\varepsilon h^{-}\max_{[x_{\frac{N}{2}-1},x_{\frac{N}{2}}]}|u''_i(\theta)| \quad \text{where } \begin{cases} x_{\frac{N}{2}} < \eta < x_{\frac{N}{2}+1}, \\ x_{\frac{N}{2}-1} < \theta < x_{\frac{N}{2}} \end{cases} \\ &\leq C\varepsilon^{-1}\sigma_1 N^{-1}\sum_{q=1}^{n}B_q(\eta) + C\varepsilon^{-1}N^{-1}\sum_{q=1}^{n}B_q(\theta) \\ &\leq C\varepsilon^{-1}\sigma_1 N^{-1} + C\varepsilon^{-1}N^{-1}B_n(\theta) \\ &\leq CN^{-1}\ln N.\end{aligned}$$

We conclude this section with the following main result which follows by using the error analysis for the regular and singular components, and the discrete maximum principle.

Theorem 6 *Let $\vec{u}$ be the solution of the continuous problem* (1), (2) *and $\vec{U}$ be the solution of the discrete problem* (13), (14). *Thus, for N sufficiently large,*

$$||(L^N(\vec{U}-\vec{u}))|| \leq CN^{-1}\ln N$$

where C is a constant independent of ε and N.

Proof. Consider the two mesh functions

$$\theta_i^{\pm}(x_j) = \begin{cases} CN^{-1}\ln N(1+2x_j) \pm \vec{L}^N(U_i(x_j)-u_i(x_j)), & j \leq \frac{N}{2} \\ CN^{-1}\ln N(d+x_j) \pm \vec{L}^N(U_i(x_j)-u_i(x_j)), & j > \frac{N}{2} \end{cases}$$

where C is suitably chosen sufficiently large constant. Hence for $j < \frac{N}{2}$, it is not hard to verify that $(\vec{\beta}^N\vec{\theta}^{\pm})_i(0) \geq \vec{0}$ and

$$\begin{aligned}(\vec{L}^N\vec{\theta}^{\pm})_i(x_j) &= C\varepsilon N^{-1}\ln N + CN^{-1}\ln N(1+2x_j)\sum_{p=1}^{n} a_{ip}(x_j) \pm \vec{L}^N(U_i(x_j)-u_i(x_j)) \\ &> CN^{-1}\ln \sum_{p=1}^{n} a_{ip}(x_j) \pm \vec{L}^N(U_i(x_j)-u_i(x_j)) \\ &> CN^{-1}\ln N\alpha \pm CN^{-1}\ln N \\ &\geq 0\end{aligned}$$

and for $j > \frac{N}{2}$,

$$\begin{aligned}(\vec{L}^N\vec{\theta}^{\pm})_i(x_j) &= C\varepsilon N^{-1}\ln N + CN^{-1}\ln N(d+x_j)\sum_{p=1}^{n} a_{ip}(x_j) \pm \vec{L}^N(U_i(x_j)-u_i(x_j)) \\ &> CN^{-1}\ln \sum_{p=1}^{n} a_{ip}(x_j) \pm \vec{L}^N(U_i(x_j)-u_i(x_j)) \\ &> CN^{-1}\ln N\alpha \pm CN^{-1}\ln N \\ &\geq 0.\end{aligned}$$

And for $j = \frac{N}{2}$

$$\begin{aligned}(\vec{L}^N\vec{\theta}^{\pm})_i(x_{\frac{N}{2}}) &= CN^{-1}\ln N\frac{(d+x_{\frac{N}{2}}+h^+-1-2x_{\frac{N}{2}})}{h^+} - CN^{-1}\ln N\frac{(1+x_{\frac{N}{2}})-(1+x_{\frac{N}{2}}-h^-)}{h^-} \\ &= CN^{-1}\ln N\frac{(h^+-1)}{h^+} - CN^{-1}\ln N \pm CN^{-1}\ln N \\ &\leq 0.\end{aligned}$$

Thus, for N sufficiently large,

$$||\vec{U}-\vec{u}|| \leq CN^{-1}\ln N$$

which completes the proof.

8 Numerical Illustration

The numerical method proposed above is illustrated through an example presented in this section.

Example 1 *Consider the following singularly perturbed robin type initial value problems with discontinuous source terms*

$$\begin{aligned} \varepsilon u_1'(x) + (2+x)u_1(x) - u_2(x) - u_3(x) &= f_1(x), \quad x \in \Omega^- \cup \Omega^+ \\ \varepsilon u_2'(x) - u_1(x) + 4u_2(x) - u_3(x) &= f_2(x), \quad x \in \Omega^- \cup \Omega^+ \\ \varepsilon u_3'(x) - u_1(x) - u_2(x) + (4+e^x)u_3(x) &= f_3(x), \quad x \in \Omega^- \cup \Omega^+ \end{aligned}$$

with

$$\beta u_1(0) = 1, \qquad \beta u_2(0) = 1, \qquad \beta u_3(0) = 1,$$

where

$$f_1(x) = \begin{cases} 1 & for\, 0 \le x \le 0.5 \\ 1 & for\, 0.5 \le x \le 1, \end{cases} \qquad f_2(x) = \begin{cases} 3 & for\, 0 \le x \le 0.5 \\ 0.5 & for\, 0.5 \le x \le 1, \end{cases} \qquad f_3(x) = \begin{cases} 2 & for\, 0 \le x \le 0.5 \\ 1 & for\, 0.5 \le x \le 1, \end{cases}$$

The exact solution of the test example is not known. Therefore, we estimate the error for $\vec{U}$ by comparing it to the numerical solution $\tilde{\vec{U}}$ obtained on the mesh $\tilde{x}_j$ that contains the mesh points of the original and their midpoints. For different values of N and the parameter ε, we compute

$$D_\varepsilon^N = ||\vec{U} - \tilde{\vec{U}}(x_i)||_{\overline{\Omega}}.$$

The numerical solution obtained by applying the fitted mesh method (13) and (14) to the Example is shown in Figure 1. The order of convergence and the error constant are calculated and are presented in Table 1.

η	Number of mesh points N			
	72	144	288	576
0.100E+01	0.190E-01	0.104E-01	0.547E-02	0.280E-02
0.250E+00	0.389E-01	0.222E-01	0.117E-01	0.605E-02
0.625E-01	0.408E-01	0.272E-01	0.169E-01	0.101E-01
0.156E-01	0.400E-01	0.266E-01	0.166E-01	0.984E-02
0.391E-02	0.398E-01	0.264E-01	0.165E-01	0.978E-02
D^N	0.408E-01	0.272E-01	0.169E-01	0.101E-01
p^N	0.588E+00	0.683E+00	0.750E+00	
C_p^N	0.151E+01	0.151E+01	0.141E+01	0.126E+01
The order of $\vec{\varepsilon}$-uniform convergence $p^* = 0.5880695E+00$				
Computed $\vec{\varepsilon}$-uniform error constant, $C_{p^*}^N = 0.1508601E+01$				

Table 1: Maximum pointwise errors D_ε^N, D^N, p^N, p^* and $C_{p^*}^N$ generated for the example.

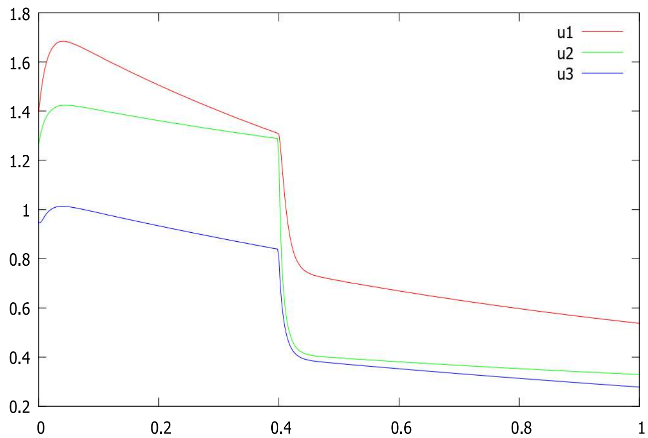

Figure 1:

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